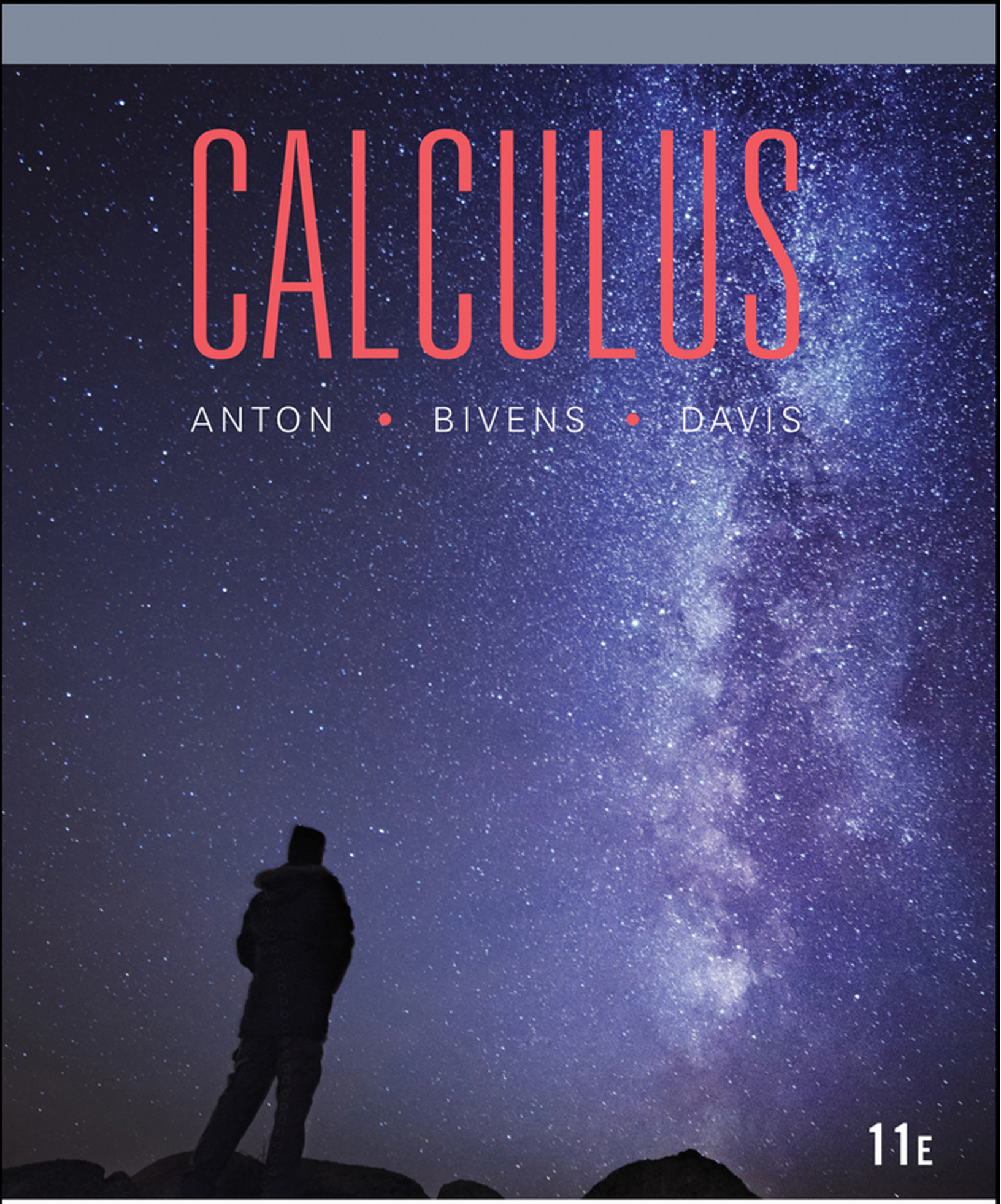
CALCULUS

ANTON • BIVENS • DAVIS



11E

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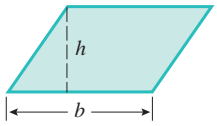
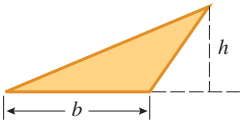
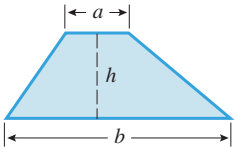
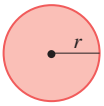
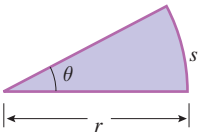
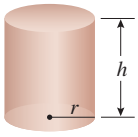
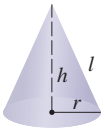
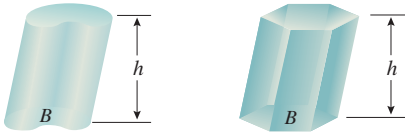
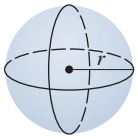
GEOMETRY FORMULAS

A = area, S = lateral surface area, V = volume, h = height, B = area of base, r = radius, l = slant height, C = circumference, s = arc length

Parallelogram	Triangle	Trapezoid	Circle	Sector
 $A = bh$	 $A = \frac{1}{2}bh$	 $A = \frac{1}{2}(a + b)h$	 $A = \pi r^2,\ C = 2\pi r$	 $A = \frac{1}{2}r^2\theta,\ s = r\theta$ (θ in radians)

Right Circular Cylinder	Right Circular Cone	Any Cylinder or Prism with Parallel Bases	Sphere
 $V = \pi r^2 h,\ S = 2\pi rh$	 $V = \frac{1}{3}\pi r^2 h,\ S = \pi rl$	 $V = Bh$	 $V = \frac{4}{3}\pi r^3,\ S = 4\pi r^2$

ALGEBRA FORMULAS

THE QUADRATIC FORMULA	THE BINOMIAL FORMULA
The solutions of the quadratic equation $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$(x+y)^n = x^n + nx^{n-1}y + \dfrac{n(n-1)}{1\cdot 2}x^{n-2}y^2 + \dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^3 + \cdots + nxy^{n-1} + y^n$ $(x-y)^n = x^n - nx^{n-1}y + \dfrac{n(n-1)}{1\cdot 2}x^{n-2}y^2 - \dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^3 + \cdots \pm nxy^{n-1} \mp y^n$

TABLE OF INTEGRALS

BASIC FUNCTIONS

1. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C$
2. $\displaystyle\int \frac{du}{u} = \ln|u| + C$
3. $\displaystyle\int e^u\,du = e^u + C$
4. $\displaystyle\int \sin u\,du = -\cos u + C$
5. $\displaystyle\int \cos u\,du = \sin u + C$
6. $\displaystyle\int \tan u\,du = \ln|\sec u| + C$
7. $\displaystyle\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1-u^2} + C$
8. $\displaystyle\int \cos^{-1} u\,du = u\cos^{-1} u - \sqrt{1-u^2} + C$
9. $\displaystyle\int \tan^{-1} u\,du = u\tan^{-1} u - \ln\sqrt{1+u^2} + C$
10. $\displaystyle\int a^u\,du = \frac{a^u}{\ln a} + C$
11. $\displaystyle\int \ln u\,du = u\ln u - u + C$
12. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$
13. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C = \ln\left|\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}u\right)\right| + C$
14. $\displaystyle\int \csc u\,du = \ln|\csc u - \cot u| + C = \ln|\tan\tfrac{1}{2}u| + C$
15. $\displaystyle\int \cot^{-1} u\,du = u\cot^{-1} u + \ln\sqrt{1+u^2} + C$
16. $\displaystyle\int \sec^{-1} u\,du = u\sec^{-1} u - \ln|u + \sqrt{u^2-1}| + C$
17. $\displaystyle\int \csc^{-1} u\,du = u\csc^{-1} u + \ln|u + \sqrt{u^2-1}| + C$

RECIPROCALS OF BASIC FUNCTIONS

18. $\displaystyle\int \frac{1}{1 \pm \sin u}\,du = \tan u \mp \sec u + C$

19. $\displaystyle\int \frac{1}{1 \pm \cos u}\,du = -\cot u \pm \csc u + C$

20. $\displaystyle\int \frac{1}{1 \pm \tan u}\,du = \tfrac{1}{2}(u \pm \ln|\cos u \pm \sin u|) + C$

21. $\displaystyle\int \frac{1}{\sin u \cos u}\,du = \ln|\tan u| + C$

22. $\displaystyle\int \frac{1}{1 \pm \cot u}\,du = \tfrac{1}{2}(u \mp \ln|\sin u \pm \cos u|) + C$

23. $\displaystyle\int \frac{1}{1 \pm \sec u}\,du = u + \cot u \mp \csc u + C$

24. $\displaystyle\int \frac{1}{1 \pm \csc u}\,du = u - \tan u \pm \sec u + C$

25. $\displaystyle\int \frac{1}{1 \pm e^u}\,du = u - \ln(1 \pm e^u) + C$

POWERS OF TRIGONOMETRIC FUNCTIONS

26. $\displaystyle\int \sin^2 u\,du = \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u + C$

27. $\displaystyle\int \cos^2 u\,du = \tfrac{1}{2}u + \tfrac{1}{4}\sin 2u + C$

28. $\displaystyle\int \tan^2 u\,du = \tan u - u + C$

29. $\displaystyle\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1} u\cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$

30. $\displaystyle\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u\sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$

31. $\displaystyle\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$

32. $\displaystyle\int \cot^2 u\,du = -\cot u - u + C$

33. $\displaystyle\int \sec^2 u\,du = \tan u + C$

34. $\displaystyle\int \csc^2 u\,du = -\cot u + C$

35. $\displaystyle\int \cot^n u\,du = -\frac{1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\,du$

36. $\displaystyle\int \sec^n u\,du = \frac{1}{n-1}\sec^{n-2} u\tan u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$

37. $\displaystyle\int \csc^n u\,du = -\frac{1}{n-1}\csc^{n-2} u\cot u + \frac{n-2}{n-1}\int \csc^{n-2} u\,du$

PRODUCTS OF TRIGONOMETRIC FUNCTIONS

38. $\displaystyle\int \sin mu\sin nu\,du = -\frac{\sin(m+n)u}{2(m+n)} + \frac{\sin(m-n)u}{2(m-n)} + C$

39. $\displaystyle\int \cos mu\cos nu\,du = \frac{\sin(m+n)u}{2(m+n)} + \frac{\sin(m-n)u}{2(m-n)} + C$

40. $\displaystyle\int \sin mu\cos nu\,du = -\frac{\cos(m+n)u}{2(m+n)} - \frac{\cos(m-n)u}{2(m-n)} + C$

41. $\displaystyle\int \sin^m u\cos^n u\,du = -\frac{\sin^{m-1} u\cos^{n+1} u}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} u\cos^n u\,du$

$$= \frac{\sin^{m+1} u\cos^{n-1} u}{m+n} + \frac{n-1}{m+n}\int \sin^m u\cos^{n-2} u\,du$$

PRODUCTS OF TRIGONOMETRIC AND EXPONENTIAL FUNCTIONS

42. $\displaystyle\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

43. $\displaystyle\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

POWERS OF u MULTIPLYING OR DIVIDING BASIC FUNCTIONS

44. $\displaystyle\int u\sin u\,du = \sin u - u\cos u + C$

45. $\displaystyle\int u\cos u\,du = \cos u + u\sin u + C$

46. $\displaystyle\int u^2\sin u\,du = 2u\sin u + (2-u^2)\cos u + C$

47. $\displaystyle\int u^2\cos u\,du = 2u\cos u + (u^2-2)\sin u + C$

48. $\displaystyle\int u^n\sin u\,du = -u^n\cos u + n\int u^{n-1}\cos u\,du$

49. $\displaystyle\int u^n\cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$

50. $\displaystyle\int u^n\ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

51. $\displaystyle\int ue^u\,du = e^u(u-1) + C$

52. $\displaystyle\int u^n e^u\,du = u^n e^u - n\int u^{n-1}e^u\,du$

53. $\displaystyle\int u^n a^u\,du = \frac{u^n a^u}{\ln a} - \frac{n}{\ln a}\int u^{n-1}a^u\,du + C$

54. $\displaystyle\int \frac{e^u\,du}{u^n} = -\frac{e^u}{(n-1)u^{n-1}} + \frac{1}{n-1}\int \frac{e^u\,du}{u^{n-1}}$

55. $\displaystyle\int \frac{a^u\,du}{u^n} = -\frac{a^u}{(n-1)u^{n-1}} + \frac{\ln a}{n-1}\int \frac{a^u\,du}{u^{n-1}}$

56. $\displaystyle\int \frac{du}{u\ln u} = \ln|\ln u| + C$

POLYNOMIALS MULTIPLYING BASIC FUNCTIONS

57. $\displaystyle\int p(u)e^{au}\,du = \frac{1}{a}p(u)e^{au} - \frac{1}{a^2}p'(u)e^{au} + \frac{1}{a^3}p''(u)e^{au} - \cdots$ [signs alternate: $+ - + - \cdots$]

58. $\displaystyle\int p(u)\sin au\,du = -\frac{1}{a}p(u)\cos au + \frac{1}{a^2}p'(u)\sin au + \frac{1}{a^3}p''(u)\cos au - \cdots$ [signs alternate in pairs after first term: $+ + - - + + - - \cdots$]

59. $\displaystyle\int p(u)\cos au\,du = \frac{1}{a}p(u)\sin au + \frac{1}{a^2}p'(u)\cos au - \frac{1}{a^3}p''(u)\sin au - \cdots$ [signs alternate in pairs: $+ + - - + + - - \cdots$]

FOR THE STUDENT

Calculus provides a way of viewing and analyzing the physical world. As with all mathematics courses, calculus involves equations and formulas. However, if you successfully learn to use all the formulas and solve all of the problems in the text but do not master the underlying *ideas*, you will have missed the most important part of calculus. If you master these ideas, you will have a widely applicable tool that goes far beyond textbook exercises.

Before starting your studies, you may find it helpful to leaf through this text to get a general feeling for its different parts:

- The opening page of each chapter gives you an overview of what that chapter is about, and the opening page of each section within a chapter gives you an overview of what that section is about. To help you locate specific information, sections are subdivided into topics that are marked with a box like this ■.
- Each section ends with a set of exercises. The answers to most odd-numbered exercises appear in the back of the book. If you find that your answer to an exercise does not match that in the back of the book, do not assume immediately that yours is incorrect—there may be more than one way to express the answer. For example, if your answer is $\sqrt{2}/2$ and the text answer is $1/\sqrt{2}$, then both are correct since your answer can be obtained by "rationalizing" the text answer. In general, if your answer does not match that in the text, then your best first step is to look for an algebraic manipulation or a trigonometric identity that might help you determine if the two answers are equivalent. If the answer is in the form of a decimal approximation, then your answer might differ from that in the text because of a difference in the number of decimal places used in the computations.
- The section exercises include regular exercises and four special categories: *Quick Check*, *Focus on Concepts*, *True/False*, and *Writing*.
  - The *Quick Check* exercises are intended to give you quick feedback on whether you understand the key ideas in the section; they involve relatively little computation, and have answers provided at the end of the exercise set.
  - The *Focus on Concepts* exercises, as their name suggests, key in on the main ideas in the section.
  - *True/False* exercises focus on key ideas in a different way. You must decide whether the statement is true in *all possible circumstances*, in which case you would declare it to be "true," or whether there are some circumstances in which it is not true, in which case you would declare it to be "false." In each such exercise you are asked to "Explain your answer." You might do this by noting a theorem in the text that shows the statement to be true or by finding a particular example in which the statement is not true.
  - *Writing* exercises are intended to test your ability to explain mathematical ideas in words rather than relying solely on numbers and symbols. All exercises requiring writing should be answered in complete, correctly punctuated logical sentences—not with fragmented phrases and formulas.
- Each chapter ends with two additional sets of exercises: *Chapter Review Exercises*, which, as the name suggests, is a select set of exercises that provide a review of the main concepts and techniques in the chapter, and *Making Connections*, in which exercises require you to draw on and combine various ideas developed throughout the chapter.
- Your instructor may choose to incorporate technology in your calculus course. Exercises whose solution involves the use of some kind of technology are tagged with icons to alert you and your instructor. Those exercises tagged with the icon require graphing technology—either a graphing calculator or a computer program that can graph equations. Those exercises tagged with the icon require a computer algebra system (CAS) such as *Mathematica*, *Maple*, or available on some graphing calculators.
- Inside the front and back covers of the text you will find endpapers that contain useful formulas.
- Five summary appendices are included at the end of the text, surveying key definitions and facts from trigonometry, general functions, and inverse functions. More detailed versions of these appendices, as well as eleven additional appendices that supply further support and/or extend the text, are available in web versions.
- The ideas in this text were created by real people with interesting personalities and backgrounds. Pictures and biographical sketches of many of these people appear throughout the book.
- Notes in the margin are intended to clarify or comment on important points in the text.

A Word of Encouragement

As you work your way through this text you will find some ideas that you understand immediately, some that you don't understand until you have read them several times, and others that you do not seem to understand, even after several readings. Do not become discouraged—some ideas are intrinsically difficult and take time to "percolate." You may well find that a hard idea becomes clear later when you least expect it.

Web Sites for this Text

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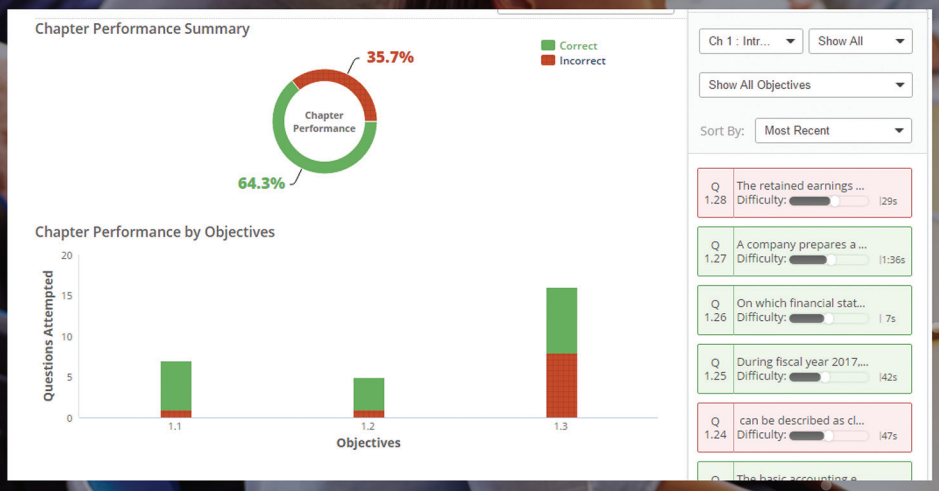

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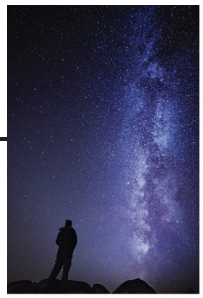

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11th EDITION

CALCULUS

LATE TRANSCENDENTALS

- **HOWARD ANTON** *Drexel University*
- **IRL BIVENS** *Davidson College*
- **STEPHEN DAVIS** *Davidson College*

WILEY

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About HOWARD ANTON

Howard Anton obtained his B.A. from Lehigh University, his M.A. from the University of Illinois, and his Ph.D. from the Polytechnic University of Brooklyn, all in mathematics. In the early 1960s he worked for Burroughs Corporation and Avco Corporation at Cape Canaveral, Florida, where he was involved with the manned space program. In 1968 he joined the Mathematics Department at Drexel University, where he taught full time until 1983. Since that time he has been an Emeritus Professor at Drexel and has devoted the majority of his time to textbook writing and activities for mathematical associations. Dr. Anton was president of the EPADEL section of the Mathematical Association of America (MAA), served on the Board of Governors of that organization, and guided the creation of the student chapters of the MAA. He has published numerous research papers in functional analysis, approximation theory, and topology, as well as pedagogical papers. He is best known for his textbooks in mathematics, which are among the most widely used in the world. There are currently more than one hundred versions of his books, including translations into Spanish, Arabic, Portuguese, Italian, Indonesian, French, Japanese, Chinese, Hebrew, and German. His textbook in linear algebra has won both the Textbook Excellence Award and the McGuffey Award from the Textbook Author's Association. For relaxation, Dr. Anton enjoys traveling and photography.

About IRL BIVENS

Irl C. Bivens, a recipient of the George Polya Award and the Merten M. Hasse Prize for Expository Writing in Mathematics, received his A.B. from Pfeiffer College and his Ph.D. from the University of North Carolina at Chapel Hill, both in mathematics. Since 1982 he has taught at Davidson College, where he currently holds the position of professor of mathematics. A typical academic year sees him teaching courses in calculus, complex analysis, and the history of mathematics. He has published numerous articles on undergraduate mathematics, as well as research papers in differential geometry. Past service to the MAA includes membership on the editorial boards of *The College Mathematics Journal,* the Problem Book series, Dolciani Mathematical Expositions, along with service on awards committees for expository writing. For recreation, he enjoys swimming, juggling, and the exploration of mathematical legerdemain.

About STEPHEN DAVIS

Stephen L. Davis received his B.A. from Lindenwood College and his Ph.D. from Rutgers University in mathematics. Having previously taught at Rutgers University and Ohio State University, Dr. Davis joined the faculty at Davidson College in 1981, where he is currently a professor of mathematics. He regularly teaches calculus, linear algebra, abstract algebra, and computer science. Professor Davis has published numerous articles on calculus reform and testing, as well as research papers on finite group theory. Professor Davis has held several offices in the Southeastern section of the MAA, including chair and secretary-treasurer, and has represented the section on the MAA Board of Governors. He has been involved in the grading of the Advanced Placement Calculus Exam since 1988 and began a four-year term as Chief Reader in 2015. He is also webmaster for the North Carolina Association of Advanced Placement Mathematics Teachers and is actively involved in nurturing mathematically talented high school students through leadership in the Charlotte Mathematics Club. For relaxation, he plays basketball, travels, and occasionally juggles. Professor Davis and his wife Elisabeth have three children, Laura, Anne, and James, all former calculus students.

To
my wife Pat and my children: Brian, David, and Lauren

In Memory of
my mother Shirley
my father Benjamin
my thesis advisor and inspiration, George Bachman
my benefactor in my time of need, Stephen Girard (1750–1831)
—HA

To
my son Robert
—IB

To
my wife Elisabeth
my children: Laura, Anne, and James
—SD

PREFACE

This eleventh edition of *Calculus* maintains those aspects of previous editions that have led to the series' success—we continue to strive for student comprehension without sacrificing mathematical accuracy, and the exercise sets are carefully constructed to avoid unhappy surprises that can derail a calculus class.

All of the changes to the eleventh edition were carefully reviewed by outstanding teachers comprised of both users and nonusers of the previous edition. The charge of this committee was to ensure that all changes did not alter those aspects of the text that attracted users of the tenth edition and at the same time provide freshness to the new edition that would attract new users.

NEW TO THIS EDITION

- Most of the pre-calculus material in the 10th edition Chapter 0 has been moved to Appendices, and the remaining Chapter 0 material is merged into Chapter 1.
- Some prose in other areas of the text has been tightened to enhance clarity and student understanding.
- New applied exercises have been added to the book and some existing applied exercises have been updated.

OTHER FEATURES

Flexibility This edition has a built-in flexibility that is designed to serve a broad spectrum of calculus philosophies—from traditional to "reform." Technology can be emphasized or not, and the order of many topics can be permuted freely to accommodate each instructor's specific needs.

Rigor The challenge of writing a good calculus book is to strike the right balance between rigor and clarity. Our goal is to present precise mathematics to the fullest extent possible in an introductory treatment. Where clarity and rigor conflict, we choose clarity; however, we believe it to be important that the student understand the difference between a careful proof and an informal argument, so we have informed the reader when the arguments being presented are informal or motivational. Theory involving ϵ-δ arguments appears in separate sections so that they can be covered or not, as preferred by the instructor.

Rule of Four The "rule of four" refers to presenting concepts from the verbal, algebraic, visual, and numerical points of view. In keeping with current pedagogical philosophy, we used this approach whenever appropriate.

Visualization This edition makes extensive use of modern computer graphics to clarify concepts and to develop the student's ability to visualize mathematical objects, particularly

those in 3-space. For those students who are working with graphing technology, there are many exercises that are designed to develop the student's ability to generate and analyze mathematical curves and surfaces.

Quick Check Exercises Each exercise set begins with approximately five exercises (answers included) that are designed to provide students with an immediate assessment of whether they have mastered key ideas from the section. They require a minimum of computation and are answered by filling in the blanks.

Focus on Concepts Exercises Each exercise set contains a clearly identified group of problems that focus on the main ideas of the section.

Technology Exercises Most sections include exercises that are designed to be solved using either a graphing calculator or a computer algebra system such as *Mathematica*, *Maple*, or the open source program *Sage*. These exercises are marked with an icon for easy identification.

Applicability of Calculus One of the primary goals of this text is to link calculus to the real world and the student's own experience. This theme is carried through in the examples and exercises.

Career Preparation This text is written at a mathematical level that will prepare students for a wide variety of careers that require a sound mathematics background, including engineering, the various sciences, and business.

Trigonometry Summary and Review Deficiencies in trigonometry plague many students, so we have included a substantial trigonometry review in Appendices A and J.

Appendix on Polynomial Equations Because many calculus students are weak in solving polynomial equations, we have included an appendix (Appendix H) that reviews the Factor Theorem, the Remainder Theorem, and procedures for finding rational roots.

Principles of Integral Evaluation The traditional Techniques of Integration is entitled "Principles of Integral Evaluation" to reflect its more modern approach to the material. The chapter emphasizes general methods and the role of technology rather than specific tricks for evaluating complicated or obscure integrals.

Historical Notes The biographies and historical notes have been a hallmark of this text from its first edition and have been maintained. All of the biographical materials have been distilled from standard sources with the goal of capturing and bringing to life for the student the personalities of history's greatest mathematicians.

Margin Notes and Warnings These appear in the margins throughout the text to clarify or expand on the text exposition or to alert the reader to some pitfall.

SUPPLEMENTS

The **Student Solutions Manual** provides detailed solutions to the odd-numbered exercises in the text. The structure of the step-by-step solutions matches those of the worked examples in the textbook. The Student Solutions Manual is also provided in digital format to students in *WileyPLUS*.

Volume I (Calculus: Early Transcendental) ISBN: 978-1-119-14781-7
Volume II (Calculus: Late Transcendental) ISBN: 978-1-119-19366-1

The Student Study Guide is available for download from the book companion Web site at www.wiley.com/college/anton or at www.howardanton.com and to users of *WileyPLUS*.

The **Instructor's Solutions Manual** contains detailed solutions to all of the exercises in the text. The Instructor's Solutions Manual is also available in PDF format on the password-protected Instructor Companion Site at www.wiley.com/college/anton or at www.howardanton.com and in *WileyPLUS*.

ISBN: 978-1-119-14782-4

The **Instructor's Manual** suggests time allocations and teaching plans for each section in the text. Most of the teaching plans contain a bulleted list of key points to emphasize. The discussion of each section concludes with a sample homework assignment. The Instructor's Manual is available in PDF format on the password-protected Instructor Companion Site at www.wiley.com/college/anton or at www.howardanton.com and in *WileyPLUS*.

The **Web Projects (Expanding the Calculus Horizon)** referenced in the text can also be downloaded from the companion Web sites and from *WileyPLUS*.

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- **Are You Ready?** quizzes gauge student mastery of chapter concepts and techniques and provide feedback on areas that require further attention.
- **ORION Algebra and Trigonometry Refresher** provides students with an opportunity to brush up on the material necessary to master calculus, as well as to determine areas that require further review.

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THE ROOTS OF CALCULUS

Today's exciting applications of calculus have roots that can be traced to the work of the Greek mathematician Archimedes, but the actual discovery of the fundamental principles of calculus was made independently by Isaac Newton (English) and Gottfried Leibniz (German) in the late seventeenth century. The work of Newton and Leibniz was motivated by four major classes of scientific and mathematical problems of the time:

- Find the tangent line to a general curve at a given point.
- Find the area of a general region, the length of a general curve, and the volume of a general solid.
- Find the maximum or minimum value of a quantity—for example, the maximum and minimum distances of a planet from the Sun, or the maximum range attainable for a projectile by varying its angle of fire.
- Given a formula for the distance traveled by a body in any specified amount of time, find the velocity and acceleration of the body at any instant. Conversely, given a formula that specifies the acceleration of velocity at any instant, find the distance traveled by the body in a specified period of time.

Newton and Leibniz found a fundamental relationship between the problem of finding a tangent line to a curve and the problem of determining the area of a region. Their realization of this connection is considered to be the "discovery of calculus." Though Newton saw how these two problems are related ten years before Leibniz did, Leibniz published his work twenty years before Newton. This situation led to a stormy debate over who was the rightful discoverer of calculus. The debate engulfed Europe for half a century, with the scientists of the European continent supporting Leibniz and those from England supporting Newton. The conflict was extremely unfortunate because Newton's inferior notation badly hampered scientific development in England, and the Continent in turn lost the benefit of Newton's discoveries in astronomy and physics for nearly fifty years. In spite of it all, Newton and Leibniz were sincere admirers of each other's work.

[*Image: Public domain image from http://commons.wikimedia.org/wiki/File:Hw-newton.jpg. Image provided courtesy of the University of Texas Libraries, The University of Texas at Austin.*]

ISAAC NEWTON (1642–1727)

Newton was born in the village of Woolsthorpe, England. His father died before he was born and his mother raised him on the family farm. As a youth he showed little evidence of his later brilliance, except for an unusual talent with mechanical devices—he apparently built a working water clock and a toy flour mill powered by a mouse. In 1661 he entered Trinity College in Cambridge with a deficiency in geometry. Fortunately, Newton caught the eye of Isaac Barrow, a gifted mathematician and teacher. Under Barrow's guidance Newton immersed himself in mathematics and science, but he graduated without any special distinction. Because the bubonic plague was spreading rapidly through London, Newton returned to his home in Woolsthorpe and stayed there during the years of 1665 and 1666. In those two momentous years the entire framework of modern science was miraculously created in Newton's mind. He discovered calculus, recognized the underlying principles of planetary motion and gravity, and determined that "white" sunlight was composed of all colors, red to violet. For whatever reasons he kept his discoveries to himself. In 1667 he returned to Cambridge to obtain his Master's degree and upon graduation became a teacher at Trinity. Then in 1669 Newton succeeded his teacher, Isaac Barrow, to the Lucasian chair of mathematics at Trinity, one of the most honored chairs of mathematics in the world.

Thereafter, brilliant discoveries flowed from Newton steadily. He formulated the law of gravitation and used it to explain the motion of the moon, the planets, and the tides; he formulated basic theories of light, thermodynamics, and hydrodynamics; and he devised and constructed the first modern reflecting telescope. Throughout his life Newton was hesitant to publish his major discoveries, revealing them only to a

select circle of friends, perhaps because of a fear of criticism or controversy. In 1687, only after intense coaxing by the astronomer, Edmond Halley (discoverer of Halley's comet), did Newton publish his masterpiece, *Philosophiae Naturalis Principia Mathematica* (The Mathematical Principles of Natural Philosophy). This work is generally considered to be the most important and influential scientific book ever written. In it Newton explained the workings of the solar system and formulated the basic laws of motion, which to this day are fundamental in engineering and physics. However, not even the pleas of his friends could convince Newton to publish his discovery of calculus. Only after Leibniz published his results did Newton relent and publish his own work on calculus.

After twenty-five years as a professor, Newton suffered depression and a nervous breakdown. He gave up research in 1695 to accept a position as warden and later master of the London mint. During the twenty-five years that he worked at the mint, he did virtually no scientific or mathematical work. He was knighted in 1705 and on his death was buried in Westminster Abbey with all the honors his country could bestow. It is interesting to note that Newton was a learned theologian who viewed the primary value of his work to be its support of the existence of God. Throughout his life he worked passionately to date biblical events by relating them to astronomical phenomena. He was so consumed with this passion that he spent years searching the Book of Daniel for clues to the end of the world and the geography of hell.

Newton described his brilliant accomplishments as follows: "I seem to have been only like a boy playing on the seashore and diverting myself in now and then finding a smoother pebble or prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."

[*Image: Public domain image from http://commons.wikimedia.org/wiki/File:Gottfried_Wilhelm_von_ Leibniz.jpg*]

GOTTFRIED WILHELM LEIBNIZ (1646–1716)

This gifted genius was one of the last people to have mastered most major fields of knowledge—an impossible accomplishment in our own era of specialization. He was an expert in law, religion, philosophy, literature, politics, geology, metaphysics, alchemy, history, and mathematics.

Leibniz was born in Leipzig, Germany. His father, a professor of moral philosophy at the University of Leipzig, died when Leibniz was six years old. The precocious boy then gained access to his father's library and began reading voraciously on a wide range of subjects, a habit that he maintained throughout his life. At age fifteen he entered the University of Leipzig as a law student and by the age of twenty received a doctorate from the University of Altdorf. Subsequently, Leibniz followed a career in law and international politics, serving as counsel to kings and princes. During his numerous foreign missions, Leibniz came in contact with outstanding mathematicians and scientists who stimulated his interest in mathematics—most notably, the physicist Christian Huygens. In mathematics Leibniz was self-taught, learning the subject by reading papers and journals. As a result of this fragmented mathematical education, Leibniz often rediscovered the results of others, and this helped to fuel the debate over the discovery of calculus.

Leibniz never married. He was moderate in his habits, quick-tempered but easily appeased, and charitable in his judgment of other people's work.

In spite of his great achievements, Leibniz never received the honors showered on Newton, and he spent his final years as a lonely embittered man. At his funeral there was one mourner, his secretary. An eyewitness stated, "He was buried more like a robber than what he really was—an ornament of his country."

1

LIMITS AND CONTINUITY

Kevin Morroun and William O'Nieal

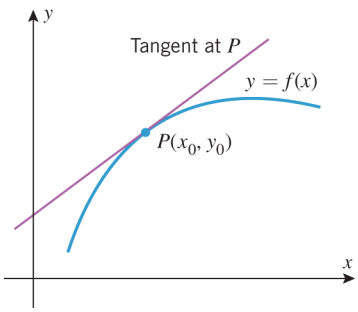
Air resistance prevents the velocity of a skydiver from increasing indefinitely. The velocity approaches a limit, called the "terminal velocity."

The development of calculus in the seventeenth century by Newton and Leibniz provided scientists with their first real understanding of what is meant by an "instantaneous rate of change" such as velocity and acceleration. Once the idea was understood conceptually, efficient computational methods followed, and science took a quantum leap forward. The fundamental building block on which rates of change rest is the concept of a "limit," an idea that is so important that all other calculus concepts are now based on it.

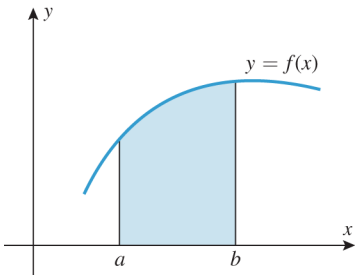
In this chapter we will develop the concept of a limit in stages, proceeding from an informal, intuitive notion to a precise mathematical definition. We will also develop theorems and procedures for calculating limits, and use limits to study "continuous" curves. We will conclude by studying continuity and other properties of trigonometric, inverse trigonometric, logarithmic, and exponential functions.

1.1 LIMITS (AN INTUITIVE APPROACH)

The concept of a "limit" is the fundamental building block on which all calculus concepts are based. In this section we will study limits informally, with the goal of developing an intuitive feel for the basic ideas. In the next three sections we will focus on computational methods and precise definitions.



▲ **Figure 1.1.1**



▲ **Figure 1.1.2**

Many of the ideas of calculus originated with the following two geometric problems:

THE TANGENT LINE PROBLEM Given a function f and a point $P(x_0, y_0)$ on its graph, find an equation of the line that is tangent to the graph at P (Figure 1.1.1).

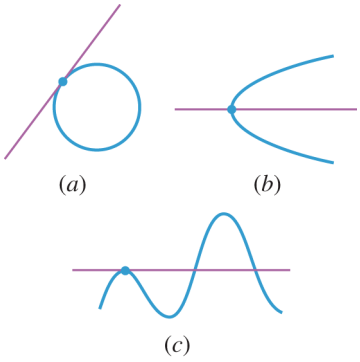
THE AREA PROBLEM Given a function f, find the area between the graph of f and an interval $[a, b]$ on the x-axis (Figure 1.1.2).

Traditionally, that portion of calculus arising from the tangent line problem is called ***differential calculus*** and that arising from the area problem is called ***integral calculus***. However, we will see later that the tangent line and area problems are so closely related that the distinction between differential and integral calculus is somewhat artificial.

TANGENT LINES AND LIMITS

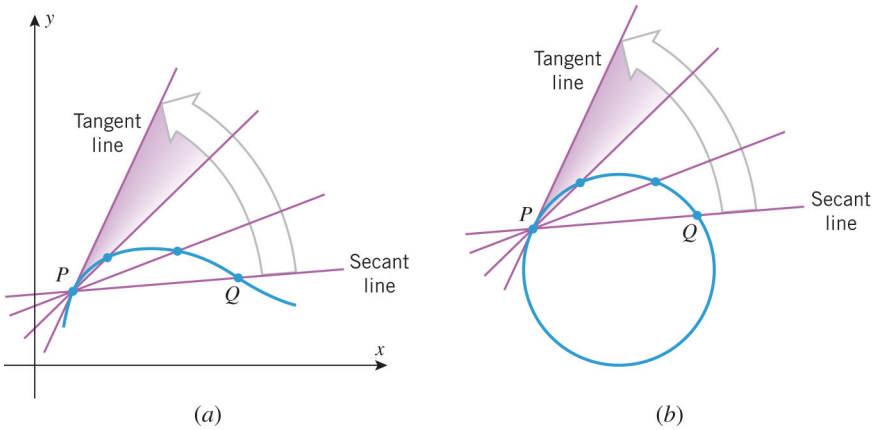
In plane geometry, a line is called *tangent* to a circle if it meets the circle at precisely one point (Figure 1.1.3*a*). Although this definition is adequate for circles, it is not appropriate for more general curves. For example, in Figure 1.1.3*b*, the line meets the curve exactly once but is obviously not what we would regard to be a tangent line; and in Figure 1.1.3*c*, the line appears to be tangent to the curve, yet it intersects the curve more than once.

To obtain a definition of a tangent line that applies to curves other than circles, we must view tangent lines another way. For this purpose, suppose that we are interested in the tangent line at a point P on a curve in the xy-plane and that Q is any point that lies on



▲ Figure 1.1.3

the curve and is different from P. The line through P and Q is called a ***secant line*** for the curve at P. Intuition suggests that if we move the point Q along the curve toward P, then the secant line will rotate toward a *limiting position*. The line in this limiting position is what we will consider to be the ***tangent line*** at P (Figure 1.1.4*a*). As suggested by Figure 1.1.4*b*, this new concept of a tangent line coincides with the traditional concept when applied to circles.



▶ Figure 1.1.4

▶ **Example 1** Find an equation for the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Solution. If we can find the slope $m_{\tan}$ of the tangent line at P, then we can use the point P and the point-slope formula for a line (Web Appendix H) to write the equation of the tangent line as

$$y - 1 = m_{\tan}(x - 1) \tag{1}$$

To find the slope $m_{\tan}$, consider the secant line through P and a point $Q(x, x^2)$ on the parabola that is distinct from P. The slope m_{sec} of this secant line is

$$m_{\text{sec}} = \frac{x^2 - 1}{x - 1} \tag{2}$$

Why are we requiring that P and Q be distinct?

Figure 1.1.4*a* suggests that if we now let Q move along the parabola, getting closer and closer to P, then the limiting position of the secant line through P and Q will coincide with that of the tangent line at P. This in turn suggests that the value of m_{sec} will get closer and closer to the value of $m_{\tan}$ as P moves toward Q along the curve. However, to say that $Q(x, x^2)$ gets closer and closer to $P(1, 1)$ is algebraically equivalent to saying that x gets closer and closer to 1. Thus, the problem of finding $m_{\tan}$ reduces to finding the "limiting value" of m_{sec} in Formula (2) as x gets closer and closer to 1 (but with $x \neq 1$ to ensure that P and Q remain distinct).

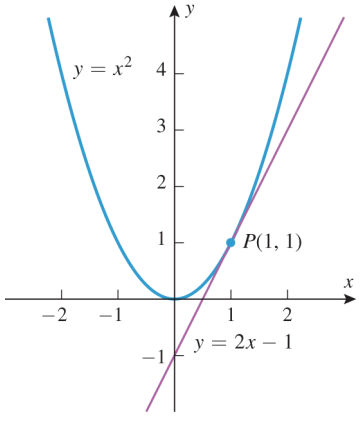
We can rewrite (2) as

$$m_{\text{sec}} = \frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{(x - 1)} = x + 1$$

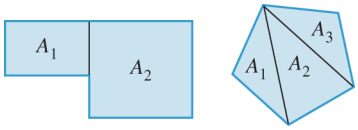
where the cancellation of the factor $(x - 1)$ is allowed because $x \neq 1$. It is now evident that m_{sec} gets closer and closer to 2 as x gets closer and closer to 1. Thus, $m_{\tan} = 2$ and (1) implies that the equation of the tangent line is

$$y - 1 = 2(x - 1) \quad \text{or equivalently} \quad y = 2x - 1$$

Figure 1.1.5 shows the graph of $y = x^2$ and this tangent line. ◀



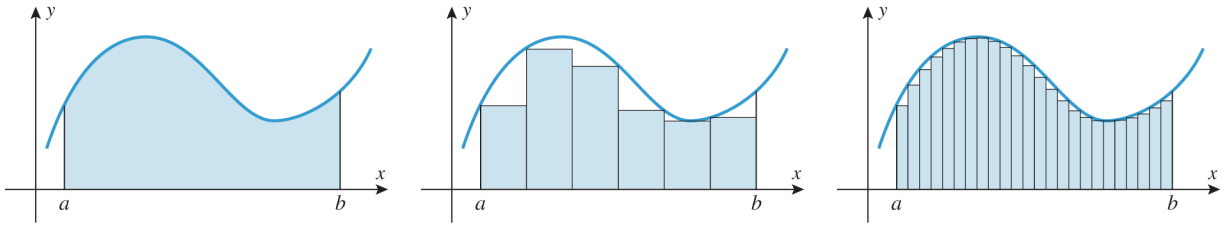
▲ Figure 1.1.5



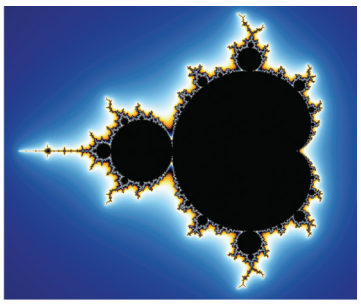
▲ Figure 1.1.6

AREAS AND LIMITS

Just as the general notion of a tangent line leads to the concept of *limit*, so does the general notion of area. For plane regions with straight-line boundaries, areas can often be calculated by subdividing the region into rectangles or triangles and adding the areas of the constituent parts (Figure 1.1.6). However, for regions with curved boundaries, such as that in Figure 1.1.7*a*, a more general approach is needed. One such approach is to begin by approximating the area of the region by inscribing a number of rectangles of equal width under the curve and adding the areas of these rectangles (Figure 1.1.7*b*). Intuition suggests that if we repeat that approximation process using more and more rectangles, then the rectangles will tend to fill in the gaps under the curve, and the approximations will get closer and closer to the exact area under the curve (Figure 1.1.7*c*). This suggests that we can define the area under the curve to be the limiting value of these approximations. This idea will be considered in detail later, but the point to note here is that once again the concept of a limit comes into play.



▲ Figure 1.1.7



Andreas Nilsson/Shutterstock

This figure shows a region called the ***Mandelbrot Set****. It illustrates how complicated a region in the plane can be and why the notion of area requires careful definition.*

DECIMALS AND LIMITS

Limits also arise in the familiar context of decimals. For example, the decimal expansion of the fraction $\frac{1}{3}$ is

$$\frac{1}{3} = 0.33333\ldots \tag{3}$$

in which the dots indicate that the digit 3 repeats indefinitely. Although you may not have thought about decimals in this way, we can write (3) as

$$\frac{1}{3} = 0.33333\ldots = 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + \cdots \tag{4}$$

which is a sum with "infinitely many" terms. As we will discuss in more detail later, we interpret (4) to mean that the succession of finite sums

$$0.3,\quad 0.3 + 0.03,\quad 0.3 + 0.03 + 0.003,\quad 0.3 + 0.03 + 0.003 + 0.0003, \ldots$$

gets closer and closer to a limiting value of $\frac{1}{3}$ as more and more terms are included. Thus, limits even occur in the familiar context of decimal representations of real numbers.

LIMITS

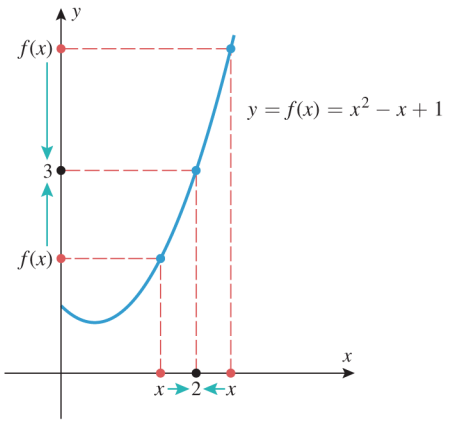
Now that we have seen how limits arise in various ways, let us focus on the limit concept itself.

The most basic use of limits is to describe how a function behaves as the independent variable approaches a given value. For example, let us examine the behavior of the function

$$f(x) = x^2 - x + 1$$

for x-values closer and closer to 2. It is evident from the graph and table in Figure 1.1.8 that the values of $f(x)$ get closer and closer to 3 as values of x are selected closer and closer to 2 on either the left or the right side of 2. We describe this by saying that the "limit of $x^2 - x + 1$ is 3 as x approaches 2 from either side," and we write

$$\lim_{x \to 2} (x^2 - x + 1) = 3 \tag{5}$$



x	1.0	1.5	1.9	1.95	1.99	1.995	1.999	2	2.001	2.005	2.01	2.05	2.1	2.5	3.0
$f(x)$	1.000000	1.750000	2.710000	2.852500	2.970100	2.985025	2.997001		3.003001	3.015025	3.030100	3.152500	3.310000	4.750000	7.000000

Left side → Right side

▲ **Figure 1.1.8**

This leads us to the following general idea.

1.1.1 LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x \to a} f(x) = L \tag{6}$$

which is read "the limit of $f(x)$ as x approaches a is L" or "$f(x)$ approaches L as x approaches a." The expression in (6) can also be written as

$$f(x) \to L \quad \text{as} \quad x \to a \tag{7}$$

Since x is required to be different from a in (6), the value of f at a, or even whether f is defined at a, has no bearing on the limit L. The limit describes the behavior of f *close to* a but not *at* a.

▶ **Example 2** Use numerical evidence to make a conjecture about the value of

$$\lim_{x \to 1} \frac{x - 1}{\sqrt{x} - 1} \tag{8}$$

Solution. Although the function

$$f(x) = \frac{x - 1}{\sqrt{x} - 1} \tag{9}$$

is undefined at $x = 1$, this has no bearing on the limit. Table 1.1.1 shows sample x-values approaching 1 from the left side and from the right side. In both cases the corresponding values of $f(x)$, calculated to six decimal places, appear to get closer and closer to 2, and hence we conjecture that

$$\lim_{x \to 1} \frac{x - 1}{\sqrt{x} - 1} = 2$$

This is consistent with the graph of f shown in Figure 1.1.9. In the next section we will show how to obtain this result algebraically. ◀

TECHNOLOGY MASTERY

Use a graphing utility to generate the graph of the equation $y = f(x)$ for the function in (9). Find a window containing $x = 1$ in which all values of $f(x)$ are within 0.5 of $y = 2$ and one in which all values of $f(x)$ are within 0.1 of $y = 2$.

Table 1.1.1

x	0.99	0.999	0.9999	0.99999		1.00001	1.0001	1.001	1.01
$f(x)$	1.994987	1.999500	1.999950	1.999995		2.000005	2.000050	2.000500	2.004988

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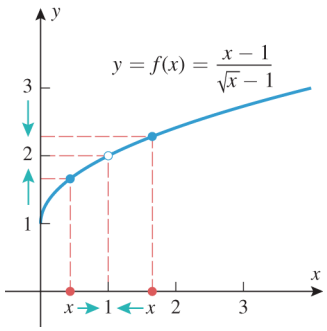


▲ **Figure 1.1.9**

Use numerical evidence to determine whether the limit in (11) changes if x is measured in degrees.

Table 1.1.2

x (RADIANS)	$y = \dfrac{\sin x}{x}$
±1.0	0.84147
±0.9	0.87036
±0.8	0.89670
±0.7	0.92031
±0.6	0.94107
±0.5	0.95885
±0.4	0.97355
±0.3	0.98507
±0.2	0.99335
±0.1	0.99833
±0.01	0.99998

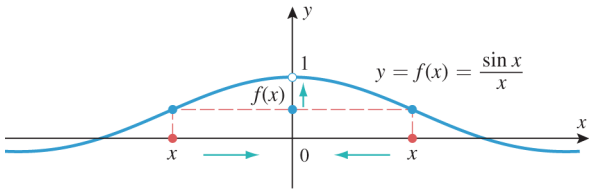
▶ **Example 3** Use numerical evidence to make a conjecture about the value of

$$\lim_{x \to 0} \frac{\sin x}{x} \tag{10}$$

Solution. With the help of a calculating utility set in radian mode, we obtain Table 1.1.2. The data in the table suggest that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \tag{11}$$

The result is consistent with the graph of $f(x) = (\sin x)/x$ shown in Figure 1.1.10. Later in this chapter we will give a geometric argument to prove that our conjecture is correct. ◀



As x approaches 0 from the left or right, $f(x)$ approaches 1.

▶ **Figure 1.1.10**

SAMPLING PITFALLS

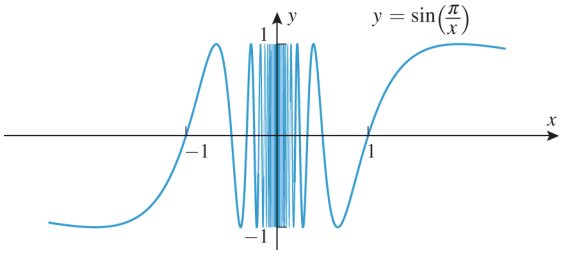
Numerical evidence can sometimes lead to incorrect conclusions about limits because of roundoff error or because the sample values chosen do not reveal the true limiting behavior. For example, one might *incorrectly* conclude from Table 1.1.3 that

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) = 0$$

The fact that this is not correct is evidenced by the graph of f in Figure 1.1.11. The graph reveals that the values of f oscillate between -1 and 1 with increasing rapidity as $x \to 0$ and hence do not approach a limit. The data in the table deceived us because the x-values selected all happened to be x-intercepts for $f(x)$. This points out the need for having alternative methods for corroborating limits conjectured from numerical evidence.

Table 1.1.3

x	$\dfrac{\pi}{x}$	$f(x) = \sin\left(\dfrac{\pi}{x}\right)$
$x = \pm 1$	$\pm\pi$	$\sin(\pm\pi) = 0$
$x = \pm 0.1$	$\pm 10\pi$	$\sin(\pm 10\pi) = 0$
$x = \pm 0.01$	$\pm 100\pi$	$\sin(\pm 100\pi) = 0$
$x = \pm 0.001$	$\pm 1000\pi$	$\sin(\pm 1000\pi) = 0$
$x = \pm 0.0001$	$\pm 10{,}000\pi$	$\sin(\pm 10{,}000\pi) = 0$
⋮	⋮	⋮

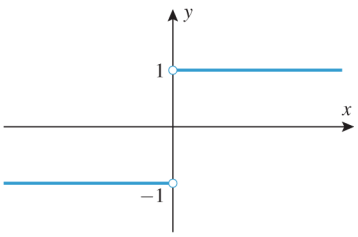


▲ **Figure 1.1.11**

ONE-SIDED LIMITS

The limit in (6) is called a ***two-sided limit*** because it requires the values of $f(x)$ to get closer and closer to L as values of x are taken from *either* side of $x = a$. However, some functions exhibit different behaviors on the two sides of an x-value a, in which case it is necessary to distinguish whether values of x near a are on the left side or on the right side of a for purposes of investigating limiting behavior. For example, consider the function

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \tag{12}$$



▲ Figure 1.1.12

which is graphed in Figure 1.1.12. As x approaches 0 from the *right*, the values of $f(x)$ approach a limit of 1 [in fact, the values of $f(x)$ are exactly 1 for all such x], and similarly, as x approaches 0 from the *left*, the values of $f(x)$ approach a limit of -1. We denote these limits by writing

$$\lim_{x\to 0^+} \frac{|x|}{x} = 1 \quad \text{and} \quad \lim_{x\to 0^-} \frac{|x|}{x} = -1 \tag{13}$$

With this notation, the superscript "+" indicates a limit from the right and the superscript "−" indicates a limit from the left.

This leads to the general idea of a ***one-sided limit***.

1.1.2 ONE-SIDED LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write

$$\lim_{x\to a^+} f(x) = L \tag{14}$$

and if the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write

$$\lim_{x\to a^-} f(x) = L \tag{15}$$

Expression (14) is read "the limit of $f(x)$ as x approaches a from the right is L" or "$f(x)$ approaches L as x approaches a from the right." Similarly, expression (15) is read "the limit of $f(x)$ as x approaches a from the left is L" or "$f(x)$ approaches L as x approaches a from the left."

As with two-sided limits, the one-sided limits in (14) and (15) can also be written as

$$f(x)\to L \quad \text{as} \quad x\to a^+$$

and

$$f(x)\to L \quad \text{as} \quad x\to a^-$$

respectively.

THE RELATIONSHIP BETWEEN ONE-SIDED LIMITS AND TWO-SIDED LIMITS

In general, there is no guarantee that a function f will have a two-sided limit at a given point a; that is, the values of $f(x)$ may not get closer and closer to any *single* real number L as $x\to a$. In this case we say that

$$\lim_{x\to a} f(x) \quad \textbf{\textit{does not exist}}$$

Similarly, the values of $f(x)$ may not get closer and closer to a single real number L as $x\to a^+$ or as $x\to a^-$. In these cases we say that

$$\lim_{x\to a^+} f(x) \quad \textbf{\textit{does not exist}}$$

or that

$$\lim_{x\to a^-} f(x) \quad \textbf{\textit{does not exist}}$$

In order for the two-sided limit of a function $f(x)$ to exist at a point a, the values of $f(x)$ must approach some real number L as x approaches a, and this number must be the same regardless of whether x approaches a from the left or the right. This suggests the following result, which we state without formal proof.

1.1.3 THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS The two-sided limit of a function $f(x)$ exists at a if and only if both of the one-sided limits exist at a and have the same value; that is,

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

▶ **Example 4** Explain why

$$\lim_{x\to 0} \frac{|x|}{x}$$

does not exist.

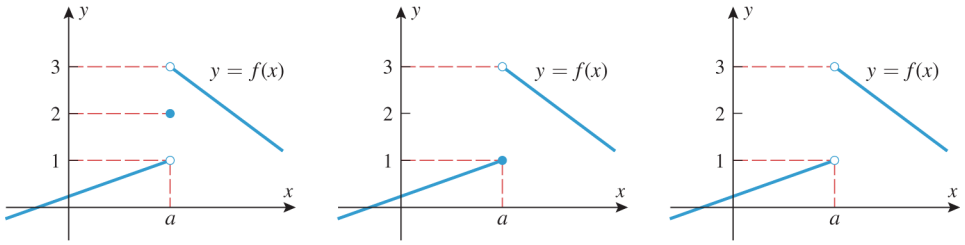
Solution. As x approaches 0, the values of $f(x) = |x|/x$ approach -1 from the left and approach 1 from the right [see (13)]. Thus, the one-sided limits at 0 are not the same. ◀

▶ Example 5 For the functions in Figure 1.1.13, find the one-sided and two-sided limits at $x = a$ if they exist.

Solution. The functions in all three figures have the same one-sided limits as $x \to a$, since the functions are identical, except at $x = a$. These limits are

$$\lim_{x \to a^+} f(x) = 3 \quad \text{and} \quad \lim_{x \to a^-} f(x) = 1$$

In all three cases the two-sided limit does not exist as $x \to a$ because the one-sided limits are not equal. ◀



▶ Figure 1.1.13

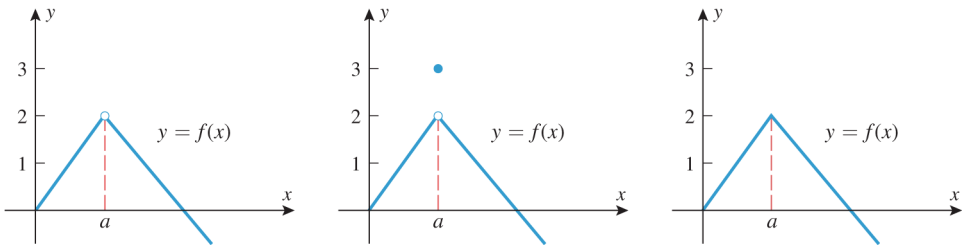
▶ Example 6 For the functions in Figure 1.1.14, find the one-sided and two-sided limits at $x = a$ if they exist.

Solution. As in the preceding example, the value of f at $x = a$ has no bearing on the limits as $x \to a$, so in all three cases we have

$$\lim_{x \to a^+} f(x) = 2 \quad \text{and} \quad \lim_{x \to a^-} f(x) = 2$$

Since the one-sided limits are equal, the two-sided limit exists and

$$\lim_{x \to a} f(x) = 2 \quad ◀$$



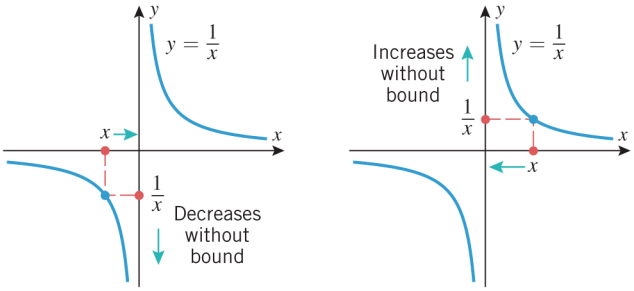
▲ Figure 1.1.14

INFINITE LIMITS

Sometimes one-sided or two-sided limits fail to exist because the values of the function increase or decrease without bound. For example, consider the behavior of $f(x) = 1/x$ for values of x near 0. It is evident from the table and graph in Figure 1.1.15 that as x-values are taken closer and closer to 0 from the right, the values of $f(x) = 1/x$ are positive and increase without bound; and as x-values are taken closer and closer to 0 from the left, the values of $f(x) = 1/x$ are negative and decrease without bound. We describe these limiting behaviors by writing

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

The symbols $+\infty$ and $-\infty$ here are *not* real numbers; they simply describe particular ways in which the limits fail to exist. Do not make the mistake of manipulating these symbols using rules of algebra. For example, it is *incorrect* to write $(+\infty) - (+\infty) = 0$.



x	-1	-0.1	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01	0.1	1
$\frac{1}{x}$	-1	-10	-100	-1000	$-10{,}000$		10,000	1000	100	10	1

Left side → Right side

▲ **Figure 1.1.15**

1.1.4 INFINITE LIMITS (AN INFORMAL VIEW) The expressions

$$\lim_{x \to a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = +\infty$$

denote that $f(x)$ increases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = +\infty$$

Similarly, the expressions

$$\lim_{x \to a^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = -\infty$$

denote that $f(x)$ decreases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = -\infty$$

▶ **Example 7** For the functions in Figure 1.1.16, describe the limits at $x = a$ in appropriate limit notation.

Solution (a). In Figure 1.1.16*a*, the function increases without bound as x approaches a from the right and decreases without bound as x approaches a from the left. Thus,

$$\lim_{x \to a^+} \frac{1}{x - a} = +\infty \quad \text{and} \quad \lim_{x \to a^-} \frac{1}{x - a} = -\infty$$

Solution (b). In Figure 1.1.16*b*, the function increases without bound as x approaches a from both the left and right. Thus,

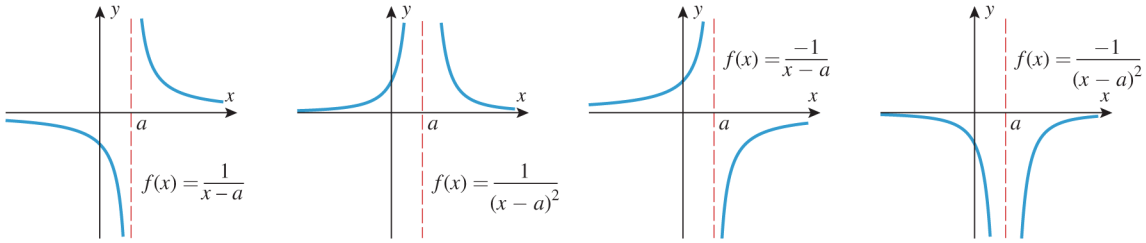
$$\lim_{x \to a} \frac{1}{(x - a)^2} = \lim_{x \to a^+} \frac{1}{(x - a)^2} = \lim_{x \to a^-} \frac{1}{(x - a)^2} = +\infty$$

Solution (c). In Figure 1.1.16*c*, the function decreases without bound as x approaches a from the right and increases without bound as x approaches a from the left. Thus,

$$\lim_{x \to a^+} \frac{-1}{x - a} = -\infty \quad \text{and} \quad \lim_{x \to a^-} \frac{-1}{x - a} = +\infty$$

Solution (d). In Figure 1.1.16*d*, the function decreases without bound as x approaches a from both the left and right. Thus,

$$\lim_{x\to a}\frac{-1}{(x-a)^2}=\lim_{x\to a^+}\frac{-1}{(x-a)^2}=\lim_{x\to a^-}\frac{-1}{(x-a)^2}=-\infty \quad \blacktriangleleft$$



(*a*) (*b*) (*c*) (*d*)

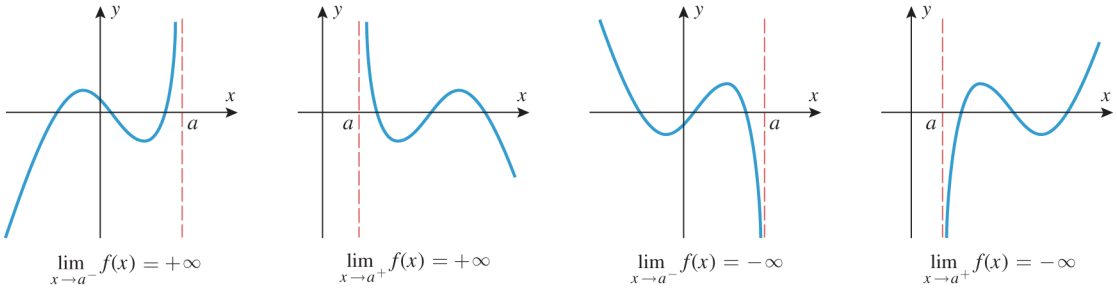
▲ **Figure 1.1.16**

VERTICAL ASYMPTOTES

Figure 1.1.17 illustrates geometrically what happens when any of the following situations occur:

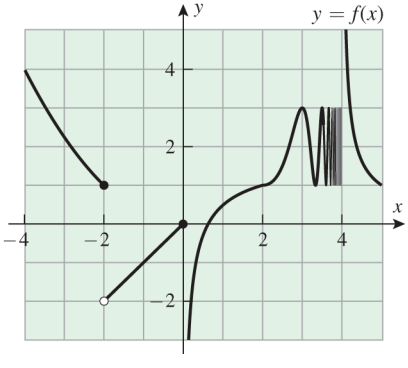
$$\lim_{x\to a^-}f(x)=+\infty,\quad \lim_{x\to a^+}f(x)=+\infty,\quad \lim_{x\to a^-}f(x)=-\infty,\quad \lim_{x\to a^+}f(x)=-\infty$$

In each case the graph of $y=f(x)$ either rises or falls without bound, squeezing closer and closer to the vertical line $x=a$ as x approaches a from the side indicated in the limit. The line $x=a$ is called a ***vertical asymptote*** of the curve $y=f(x)$ (from the Greek word *asymptotos*, meaning "nonintersecting").



▲ **Figure 1.1.17**

In general, the graph of a single function can display a wide variety of limits.



▲ **Figure 1.1.18**

► **Example 8** For the function f graphed in Figure 1.1.18, find

(a) $\lim_{x\to -2^-}f(x)$ (b) $\lim_{x\to -2^+}f(x)$ (c) $\lim_{x\to 0^-}f(x)$ (d) $\lim_{x\to 0^+}f(x)$
(e) $\lim_{x\to 4^-}f(x)$ (f) $\lim_{x\to 4^+}f(x)$ (g) the vertical asymptotes of the graph of f.

Solution (a) and (b).

$$\lim_{x\to -2^-}f(x)=1=f(-2) \quad \text{and} \quad \lim_{x\to -2^+}f(x)=-2$$

Solution (c) and (d).

$$\lim_{x\to 0^-}f(x)=0=f(0) \quad \text{and} \quad \lim_{x\to 0^+}f(x)=-\infty$$

Solution (e) and (f).

$$\lim_{x\to 4^-}f(x) \text{ does not exist due to oscillation} \quad \text{and} \quad \lim_{x\to 4^+}f(x)=+\infty$$

Solution (g). The y-axis and the line $x=4$ are vertical asymptotes for the graph of f. ◄